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# On special Bäcklund autotransformations

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**Abstract.** All Klein-Gordon, $\omega_{xy}=f(\omega)$, and evolution, $\omega_t=f(\omega,\omega_x,\ldots,\omega_{x\ldots x})$, equations admitting autotransformations $\varphi=a[\omega]$ are found ($a[\omega]$ is such a function of $\omega$ and its finite-order derivatives that any solution $\omega$ is mapped into a solution $\varphi$ of the same equation). Besides the linear ones, they include only the Liouville equation and the Burgers equation hierarchy. Their special Bäcklund autotransformations $\varphi=a[\omega]$ are also found.

## 1. Introduction

Some of the Bäcklund transformations of partial differential equations (PDE), such as the well known Hopf-Cole [1, 2] $\varphi=\omega^{-1}\omega_x$ and Miura [3] $\varphi=\omega_x-\frac{1}{2}\omega^2$ transformations, have the special form $\varphi=a[\omega]$, where $a[\omega]$ is a differential function of $\omega$, i.e. a function of $\omega$ and its finite-order derivatives. An equation $\varphi=a[\omega]$, which determines such a differential substitution, is not necessarily an ordinary differential or first-order equation, e.g. the map $\varphi=\ln(2\omega_x\omega_y\omega^{-2})$ of $\omega_{xy}=0$ into the Liouville equation $\varphi_{xy}=\exp\varphi$ [4] or the maps $\varphi=\omega_{xx}-\frac{1}{2}\omega_x^2$ and $\varphi=\omega_x^{-1}\omega_{xxx}-\frac{3}{2}\omega_x^{-2}\omega_{xx}^2$ of certain continual classes of evolution equations [5, 6]. A differential substitution $\varphi=a[\omega]$ maps *any* solution $\omega$ of a PDE $\mathrm{d}[\omega]=0$ into a solution $\varphi$ of another PDE $\mathrm{d}'[\varphi]=0$; as a rule these PDE are *different.* In contrast, a Bäcklund autotransformation, which connects a solution $\omega$ of a PDE $\mathrm{d}[\omega]=0$ with another solution $\varphi$ of *the same* PDE $\mathrm{d}[\varphi]=0$, contains derivatives of *both* solutions as a rule [4, 7-10].

In this paper, we deal with Bäcklund transformations, which are differential substitutions and Bäcklund autotransformations at the same time. An equation $\varphi=a[\omega]$ is said to be a special Bäcklund autotransformation (SBA) of a PDE $\mathrm{d}[\omega]=0$ if it maps *any* solution $\omega$ of the PDE into a solution $\varphi$ of the same PDE, i.e. $\mathrm{d}[a[\omega]]=0$ if $\mathrm{d}[\omega]=0$. If a PDE admits one SBA $\varphi=a[\omega]$, then it admits *at least* the countable set of SBA: $\varphi=a[\omega]$, $\varphi=a[a[\omega]]$, $\varphi=a[a[a[\omega]]]$, etc. However, this interesting property is not the main reason why SBA need to be studied. SBA were very rarely mentioned in the literature. Only the first-order SBA [8], the countable set of SBA [11] of the Burgers equation, and the countable set of SBA [12] of the third-order Burgers equation are known. Why? There was no systematical study of SBA, of course, but the reason is quite different. It appears that very few PDE admit SBA. In this paper, we search through two infinite classes of PDE for those admitting SBA but find only the Liouville equation and the hierarchy of the Burgers equation besides linear PDE. The investigated classes are one-component two-dimensional Klein-Gordon equations $\omega_{xy}=f(\omega)$ with arbitrary $f(\omega)$ (see section 2 of this paper) and one-component (1+1)-dimensional local evolution equations $\omega_t = f(\omega, \omega_x, \ldots, \omega_{x\ldots x})$ with arbitrary $f[\omega]$ of any order (section 3). The corresponding SBA are also found. The results are discussed in section 4.

## 2. Klein–Gordon equations

Let us consider a Klein–Gordon equation (KGE) $\omega_{xy} = f(\omega)$. By means of the equation and its differential prolongations, we express all differential functions of a solution $\omega$ in terms of independent differential variables $\omega, \omega_1, \omega_2, \ldots, \omega_{\bar{1}}, \omega_{\bar{2}}, \ldots$ ($\omega_k = \partial^k\omega/\partial x^k$, $\omega_{\bar{k}} = \partial^k\omega/\partial y^k$). Total derivatives of differential functions (followed by the reduction to independent differential variables if necessary) are denoted as $\partial_x$ and $\partial_y$. According to the definition of SBA, $\varphi = a(\omega, \omega_1, \ldots, \omega_m, \omega_{\bar{1}}, \ldots, \omega_{\bar{n}})$ is a SBA of KGE $\omega_{xy} = f(\omega)$ if and only if

$$\partial_x\partial_y a[\omega] \equiv f(a[\omega]) \tag{1}$$

where $\equiv$ indicates that (1) must be satisfied identically with *any function* $\omega$. Indeed, since (1) must be satisfied with any solution $\omega$, it must be a differential prolongation of the KGE [13], but such prolongations, being expressed in terms of independent differential variables, become identities. To find $f(\omega)$ and $a[\omega]$ we differentiate (1) with respect to $\omega_k$ and $\omega_{\bar{k}}$ and thus get new identities which simplify the analysis of the original one.

Calculating $\partial/\partial\omega_{m+1}$ and $\partial/\partial\omega_{\overline{n+1}}$ of (1), we get

$$\partial_y a_{\omega_m} \equiv \partial_x a_{\omega_{\bar{n}}} \equiv 0 \qquad (a_{\omega_m} = \partial a/\partial\omega_m,\ a_{\omega_{\bar{n}}} = \partial a/\partial\omega_{\bar{n}}).$$

Therefore $a_{\omega_m}$ and $a_{\omega_{\bar{n}}}$ are constants according to the Zhiber–Shabat lemma [14] if $f\,\mathrm{d}^2f/\mathrm{d}\omega^2 \neq (\mathrm{d}f/\mathrm{d}\omega)^2$. In this case, $\partial^2/\partial\omega_m^2$ of (1) gives $\mathrm{d}^2f/\mathrm{d}\omega^2 = 0$ but $\mathrm{d}f/\mathrm{d}\omega \neq 0$. Proceeding with differentiation of (1) with respect to $\omega_k$ and $\omega_{\bar{k}}$, we get

$$f(\omega) = \xi\omega + \eta$$
$$a[\omega] = \xi^{-1}\eta(\lambda - 1) + \lambda\omega + \sum_{k=1}^{m} \mu_k\omega_k + \sum_{k=1}^{n} \nu_k\omega_{\bar{k}} \tag{2}$$

where constants $\xi$, $\eta$, $\lambda$, $\mu_k$, $\nu_k$ ($\xi \neq 0$) and orders $m$, $n$ are arbitrary. Now let $f\,\mathrm{d}^2f/\mathrm{d}\omega^2 = (\mathrm{d}f/\mathrm{d}\omega)^2$, i.e. $f(\omega) = 0$, $\xi$, $\xi\exp(\eta\omega)$. The first two special cases are linear, and in the same way we get

$$f(\omega) = 0 \qquad a[\omega] = \lambda\omega + p(\omega_1, \ldots, \omega_m) + q(\omega_{\bar{1}}, \ldots, \omega_{\bar{n}}) \tag{3}$$

$$f(\omega) = \xi \qquad a[\omega] = \omega + \mu\omega_1 + \nu\omega_{\bar{1}} + p(\omega_2, \ldots, \omega_m) + q(\omega_{\bar{2}}, \ldots, \omega_{\bar{n}}) \tag{4}$$

where constants $\lambda$, $\xi$, $\mu$, $\nu$ ($\xi \neq 0$), functions $p$, $q$ and orders $m$, $n$ are arbitrary. The last special case $\omega_{xy} = \xi\exp(\eta\omega)$ is the Liouville equation with exact linearization $\omega = \eta^{-1}\ln(2\xi^{-1}\eta^{-1}\psi_x\psi_y\psi^{-2})$, $\psi_{xy} = 0$ [4]. In this case, the following nonlinear variables $\omega, \omega_1, \alpha, \alpha_1, \alpha_2, \ldots, \omega_{\bar{1}}, \beta, \beta_{\bar{1}}, \beta_{\bar{2}}, \ldots$ ($\alpha = \omega_2 - \frac{1}{2}\eta\omega_1^2$, $\beta = \omega_{\bar{2}} - \frac{1}{2}\eta\omega_{\bar{1}}^2$, $\alpha_k = \partial_x^k\alpha$, $\beta_{\bar{k}} = \partial_y^k\beta$) [14] simplify the analysis of (1) very much because $\partial_y\alpha \equiv \partial_x\beta \equiv 0$. Differentiating (1) with respect to independent differential variables $\omega$, $\omega_1$, $\omega_{\bar{1}}$, $\alpha_k$, $\beta_{\bar{k}}$, we get a set of new identities which can be analysed easily. Then the Liouville equation turns out to admit four continual classes of SBA:

$$
\begin{aligned}
&f(\omega)=\xi \exp(\eta\omega)\\
&a^{(1)}[\omega]=\eta^{-1}\ln\left(\frac{2\partial_x u\partial_y v}{\xi\eta(u+v)^2}\right)\\
&a^{(2)}[\omega]=\omega+\eta^{-1}\ln\left(\frac{(\partial_x u+\frac{1}{2}\eta u^2+\alpha)(\partial_y v+\frac{1}{2}\eta v^2+\beta)}{[\frac{1}{2}\eta(\omega_1+u)(\omega_{\bar{1}}+v)-\xi\exp(\eta\omega)]^2}\right)\\
&a^{(3)}[\omega]=\omega+\eta^{-1}\ln\left(\frac{2(\partial_x u+\frac{1}{2}\eta u^2+\alpha)}{\eta(\omega_1+u)^2}\right)\\
&a^{(4)}[\omega]=\omega+\eta^{-1}\ln\left(\frac{2(\partial_y v+\frac{1}{2}\eta v^2+\beta)}{\eta(\omega_{\bar{1}}+v)^2}\right)
\end{aligned}
\tag{5}
$$

where constants $\xi$, $\eta$ ($\xi\eta\neq 0$), functions $u(\alpha,\ldots,\alpha_{m-3})$, $v(\beta,\ldots,\beta_{\overline{n-3}})$ and orders $m$, $n$ are arbitrary.

## 3. Local evolution equations

Considering local evolution equations (LEE) $\omega_t=f(\omega,\omega_1,\ldots,\omega_p)$ ($p\geqslant 1, \omega_k=\partial^k\omega/\partial x^k$), we choose independent differential variables to be $\omega$, $\omega_1$, $\omega_2$, $\omega_3$, etc. A LEE $\omega_t=f[\omega]$ admits a SBA $\varphi=a(\omega,\omega_1,\ldots,\omega_n)$ ($n\geqslant 1$) if and only if the differential identity

$$
f[a[\omega]]\equiv\sum_{k=0}^{n} a_{\omega_k}[\omega]\partial_x^k f[\omega] \tag{6}
$$

is satisfied with *any function* $\omega$ ($\partial_x=\Sigma_k\,\omega_{k+1}\partial/\partial\omega_k$, $\partial_x^k=\partial_x\partial_x^{k-1}$, $\partial_x^0=1$, $\omega_0=\omega$). We have to find all differential functions $f[\omega]$ and $a[\omega]$ which satisfy (6). Problems like this are usually solved via the Faà de Bruno formula [4, 12]. We propose another approach based on the following lemma:

$$
\frac{\partial}{\partial\omega_r}\partial_x^s\equiv\sum_k\binom{s}{k}\partial_x^k\frac{\partial}{\partial\omega_{r-s+k}}
$$

where binomial coefficients $\binom{s}{k}$ equal $s!/k!(s-k)!$ at $0\leqslant k\leqslant s$ and zero at $k<0$ and $k>s$, orders $r$ and $s$ are arbitrary, and the sum is taken over all non-zero items. (Proof: induction by $s$.) By means of this lemma, we can differentiate (6) with respect to independent differential variables and thus get simpler identities. From $\partial/\partial\omega_{p+n}$ of (6), we get $f_{\omega_p}[a[\omega]]\equiv f_{\omega_p}[\omega]$. *Since $n\geqslant 1$, the separant of any* LEE *sought is constant,* $f_{\omega_p}=\xi_p$.

Let $p=1$, then $\omega_t=\xi_1\omega_1+g(\omega)$. If $g(\omega)=0$, then (6) is satisfied, and we have

$$
f[\omega]=\xi_1\omega_1 \qquad a[\omega]=a(\omega,\omega_1,\ldots,\omega_n) \tag{7}
$$

with arbitrary constant $\xi_1$, function $a$ and order $n$. If $g(\omega)\neq 0$, a suitable point transformation $\omega\rightarrow\psi(\omega)$ changes $g(\omega)$ into a constant, and (6) gives

$$
f[\omega]=\xi_1\omega_1+\eta \qquad a[\omega]=\omega+b(\omega_1,\omega_2,\ldots,\omega_n) \tag{8}
$$

with arbitrary constants $\xi_1$, $\eta$, function $b$ and order $n$.

Now let $p>1$; then $\partial/\partial\omega_{p+n-1}$ of (6) gives

$$
f_{\omega_{p-1}}[a[\omega]]-f_{\omega_{p-1}}[\omega]\equiv -p\xi_p\partial_x\ln a_{\omega_n}[\omega] \tag{9}
$$

i.e. it is necessary that $f_{\omega_{p-1}}[\omega]=g(\omega)\omega_1+h(\omega)$ where $g(\omega)$ can be made zero via suitable $\omega\to\psi(\omega)$. Now, $\partial/\partial\omega_{n+1}$ of (9) gives $a_{\omega_n\omega_n}=0$, and $\partial^2/\partial\omega_n^2$ of (9) gives $f_{\omega_{p-1}}=\sigma\omega+\xi_{p-1}$ with arbitrary constants $\sigma$ and $\xi_{p-1}$. If $\sigma=0$, $a_{\omega_n}$ is constant due to (9). Then, differentiating (6) with respect to $\omega_{p+n-2},\ldots,\omega$ and obtaining new identities like (9), we have

$$f[\omega]=\sum_{k=0}^{p}\xi_k\omega_k+\eta \qquad a[\omega]=\sum_{k=0}^{n}\mu_k\omega_k+\nu \tag{10}$$

with arbitrary constants $\xi_k$, $\eta$, $\mu_k$, $\nu$: $\xi_0\nu=\eta(\mu_0-1)$, $p\geqslant 2$, $n\geqslant 1$. If $\sigma\neq 0$, we can make $\sigma=p\xi_p$ via a suitable scale transformation of $\omega$; (9) gives $a[\omega]=\omega+\partial_x\ln(\omega_{n-1}+b)$ with yet unknown $b(\omega,\ldots,\omega_{n-2})$. Making use of the operator $\partial_x+\omega$ and differentiating (6) with respect to $\omega_{p+n-2},\ldots,\omega$, we have

$$\begin{aligned} f[\omega]&=\partial_x\left(\sum_{k=0}^{p}\xi_k(\partial_x+\omega)^k 1\right)\\ a[\omega]&=\omega+\partial_x\ln\left(\sum_{k=0}^{n}\mu_k(\partial_x+\omega)^k 1\right)\end{aligned} \tag{11}$$

with arbitrary constants $\xi_k$ and $\mu_k$: $p\geqslant 2$, $n\geqslant 1$. These LEE and SBA change into (10) with $\eta=\nu=0$ via the Hopf-Cole transformation $\omega\to\omega^{-1}\omega_x$ and $\varphi\to\varphi^{-1}\varphi_x$; LEE (11) are the hierarchy of the Burgers equation with the recursion operator $\partial_x+\omega+\omega_x\partial_x^{-1}$ [4].

## 4. Conclusion

We have found all KGE and LEE admitting SBA, and also all those particular SBA. The results are listed in (2)-(5) for KGE exactly and in (7), (8), (10) and (11) for LEE up to $\omega\to\psi(\omega)$ and $\varphi\to\psi(\varphi)$ with any $\psi$. All those PDE are either linear or exactly linearizable via point and Bäcklund transformations. The results show why most of the known Bäcklund autotransformations contain derivatives of both solutions. Moreover, SBA still remain the only known Bäcklund autotransformations $b[\varphi,\omega]=0$ in which the highest-order derivatives of $\varphi$ and $\omega$ are different.

Any PDE has either no SBA or an infinite number of SBA, but those infinities are quite different. Namely, exactly solvable PDE (of which general solutions are some fixed differential functions of arbitrary functions) have continual classes of SBA (3), (5), (7) and (8), while PDE, solvable via Fourier transformation after necessary linearization, have discrete classes of many-parameter SBA (2), (10) and (11). The same is true for Lie-Bäcklund algebras (LBA) [4]: LBA of exactly solvable PDE have continual bases, while LBA of Fourier-solvable PDE have discrete bases. We cannot explain this correlation because there is no effective theory of LBA with non-local elements at present; namely non-local infinitesimal symmetries correspond to such local finite transformations as SBA.

Certainly, SBA (2)-(4), (7), (8) and (10) of linear PDE are quite trivial, and SBA (11) of the hierarchy of the Burgers equation can be derived from those of corresponding linear LEE [8, 11]. It is, however, not so easy to explain SBA (5) in terms of the well known linearization of the Liouville equation. Even if SBA (5) are of minor importance to applications, the results indicate that such a highly symmetrical PDE as the Liouville equation [4, 14] can admit *not only one* Bäcklund autotransformation [15]. Exactly solvable many-component Liouville equations [16] can also be expected to admit continual classes of SBA.

The Miura transformation is known to connect very wide (continual) classes of LEE [5]. However the results of section 3 indicate that $\omega_t = \xi\omega_x$ is the only LEE which is mapped into itself via the Miura transformation. The same is true for other differential substitutions mentioned in the introduction.

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